Time evolution of the extremely diluted Blume-Emery-Griffiths neural network

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A study of the time evolution and a stability analysis of the phases in the extremely diluted Blume-Emery-Griffiths neural network model are shown to yield new phase diagrams in which fluctuation retrieval may drive pattern retrieval. It is shown that saddle-point solutions associated with fluctuation overlaps slow down the flow of the network states towards the retrieval fixed points. A comparison of the performance with other three-state networks is also presented.

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A novel idea suggested recently in the theory of attractor neural networks is to use information theory to infer the learning rule of an optimally performing three-state network [1]. Optimal means that although the network might start initially far from the embedded pattern, i.e., having a vanishingly small initial mutual information, it is still able to retrieve it. The study of this mutual information leads to Blume-Emery-Griffiths-type (BEG) network models with Hebbian-like learning rules [1,2]. Its structure also reveals that the retrieval overlap and fluctuation overlap are the relevant order parameters in order to study the network performance.

It has been argued in Ref. [1] that in an extremely diluted architecture of the BEG-type new states associated with the fluctuation overlap, the quadrupolar states Q appear for all values of the synaptic noise (temperature T). However, neither the stability of these states nor their time evolution have been discussed in detail. These are precisely the subjects of this Brief Report. In particular, we find that due to the presence of long transients in the dynamic evolution of the network, we need a finite activity dependent threshold in order to stabilize these Q states and, hence, part of the phase diagrams are altered in a substantial way. Moreover, we clarify the explicit role of the fluctuation overlap in enhancing the retrieval performance of the network compared with other three-state networks. This study further allows us to advocate the use of these Q states as new information carriers in practical applications, e.g., in pattern recognition where, looking at a black and white picture on a gray background, such a state would tell us the exact location of the picture with respect to the background without finding the details of the picture itself. Whether these states could also model such retrieval focusing problems discussed in the framework of cognitive neuroscience (see, e.g., Ref. [3]) is an interesting thought.

Consider a three-state network with symmetrically distributed neuron states $\sigma_{i,t}=0,\pm 1$ on sites $i=1,\ldots,N$, at time step t, where $\sigma_{i,t}=\pm 1$ denote the active states. A set of p ternary patterns, $\{\xi_i^\mu=0,\pm 1\}$, $\mu=1,\ldots,p$, where $\xi_i^\mu=\pm 1$ are the active ones, assumed to be independent random variables following the probability distribution

$$p(\xi_i^\mu)=a\,\delta(|\xi_i^\mu|^2-1)+(1-a)\,\delta(\xi_i^\mu)\,, \tag{1}$$

are stored in the network. Hence, the mean $\langle \xi_i^\mu \rangle=0$ and the variance $a=\langle (\xi_i^\mu)^2\rangle$ is the activity of the patterns.

At each time step we regard the patterns as the inputs and the neuron states as the outputs of the network. Then we can consider the mutual information [4,5] between neurons and patterns $I^\mu(\sigma_t,\xi^\mu)$, which can be expressed in terms of the order parameters of the network [6]. Provided the neural activity $q_t=\langle\langle \sigma_t^2\rangle_{\sigma|\xi}\rangle_\xi \sim a$, the initial mutual information becomes to leading order [1],

$$I^\mu(\sigma_0,\xi^\mu)\approx \tfrac{1}{2}(m_0^\mu)^2+\tfrac{1}{2}(l_0^\mu)^2, \tag{2}$$

for vanishingly small retrieval overlap $m_t^\mu=\langle\langle \sigma_t\rangle_{\sigma|\xi}\xi^\mu/a\rangle_\xi$ between σ_t and ξ_i^μ, and vanishingly small fluctuation overlap $l_t^\mu=\langle\langle \sigma_t^2\rangle_{\sigma|\xi}\eta^\mu\rangle_\xi$ between σ_t^2 and the normalized pattern fluctuations $\eta_i^\mu=((\xi_i^\mu)^2-a)/a(1-a)$ both at $t=0$.

Thus, m_t^μ and l_t^μ constitute the minimal set of overlaps needed to describe the evolution of the mutual information, instead of just m_t^μ for the three-state Ising network [7,8]. The initial quadratic form suggests a learning rule that consists of two Hebbian-like parts [1,2], $J_{ij}=(1/a^2N)\Sigma_\mu \xi_i^\mu \xi_j^\mu$ and $K_{ij}=(1/N)\Sigma_\mu \eta_i^\mu \eta_j^\mu$. Since we are considering the extremely diluted version of the model both Hebbian weights are multiplied with a factor $C_{ij}N/C$ where C_{ij} is a random variable assuming values 0 and 1, with mean $C\sim O(\ln N/N)$ [8,9]. The first part is the usual rule that accounts for cooperation between the same type of active (inactive) patterns on different neurons, while the second one is a novel part that favors the cooperation between active (inactive) patterns regardless of their type. So, one could still have some kind of recognition when the main pattern recognition fails.

The parallel stochastic dynamics for this model ruled by the state-flip probability

$$p(\sigma_{i,t+1}|\{\sigma_{i,t}\})=\exp[-\beta(h_{i,t}\sigma_{i,t}+\theta_{i,t}\sigma_{i,t}^2)]/Z_{i,t}\,, \tag{3}$$

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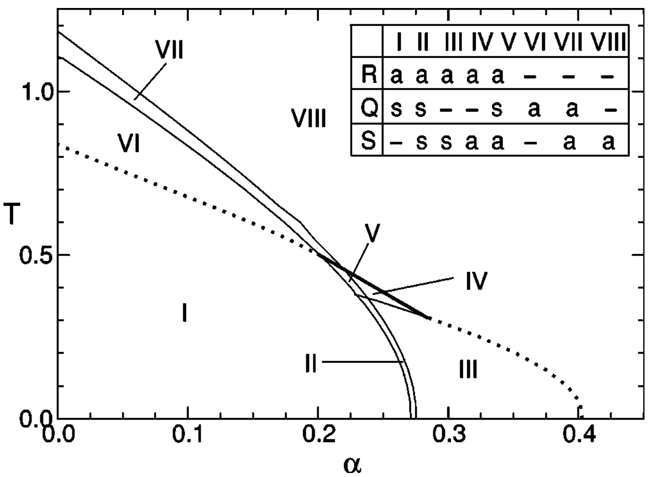

FIG. 1. The (T,α) phase diagram for the extremely diluted BEG network with pattern activity $a=0.8$. The meaning of the lines and table is explained in the text.

$$h_{i,t}=\sum_j J_{ij}\sigma_{j,t}, \quad \theta_{i,t}=\sum_j K_{ij}\sigma_{j,t}^2, \tag{4}$$

$$Z_{i,t}=1+2e^{\beta\theta_{i,t}}\cosh(\beta h_{i,t}), \tag{5}$$

with $\beta=a/T$ can be solved exactly (there are no feedback loops) in the limit $N,C\to\infty$ in a standard way [1,8,9]. Assuming one condensed pattern, one obtains [1]

$$m_{t+1}=\int Dy\int Dz \;\; F_\beta\left(\frac{m_t}{a}+y\Delta_t;\frac{l_t}{a}+z\frac{\Delta_t}{1-a}\right), \tag{6}$$

$$n_{t+1}=\int Dy\int Dz \;\; G_\beta\left(\frac{m_t}{a}+y\Delta_t;\frac{l_t}{a}+z\frac{\Delta_t}{1-a}\right), \tag{7}$$

$$s_{t+1}=\int Dy\int Dz \;\; G_\beta\left(y\Delta_t;-\frac{l_t}{1-a}+z\frac{\Delta_t}{1-a}\right), \tag{8}$$

together with the dynamic activity $q_t=an_t+(1-a)s_t$. Here n_t is the activity overlap $n_t^\mu=\langle\langle\sigma_t^2\rangle_{\sigma|\xi}(\xi^\mu)^2/a\rangle_\xi$ for the condensed pattern and $l_t=(n_t-q_t)/(1-a)$. As usual, $Dx=\exp(-x^2/2)dx/\sqrt{2\pi}$, while

$$F_\beta(h_t,\theta_t)=2e^{\beta\theta_t}\sinh(\beta h_t)/Z_t, \tag{9}$$

$$G_\beta(h_t,\theta_t)=2e^{\beta\theta_t}\cosh(\beta h_t)/Z_t \tag{10}$$

are the stochastic transfer functions that determine the thermal averages of σ_i and $(\sigma_i)^2$, while $\Delta_t^2=\alpha q_t/a^2$ and $\Delta_t^2/(1-a)^2$ are the variances in the Gaussian local fields h and θ, respectively. In all these expressions we have left out the effective single-site index i. The time evolution of the information becomes $i_t=I_t\alpha$, for the storage ratio $\alpha=p/C$.

The stationary states of the network dynamics (6)–(10) are shown in Fig. 1 for a typical activity of $a=0.8$ and $q\sim a$. In addition to the retrieval and quadrupolar phases, $R(m\neq0,l\neq0)$ and $Q(m=0,l\neq0)$, there is a self-sustained activity phase $S(m=0,l=0)$, also referred to as the zero phase Z [1,11]. Stable states are attractors (a) and there are

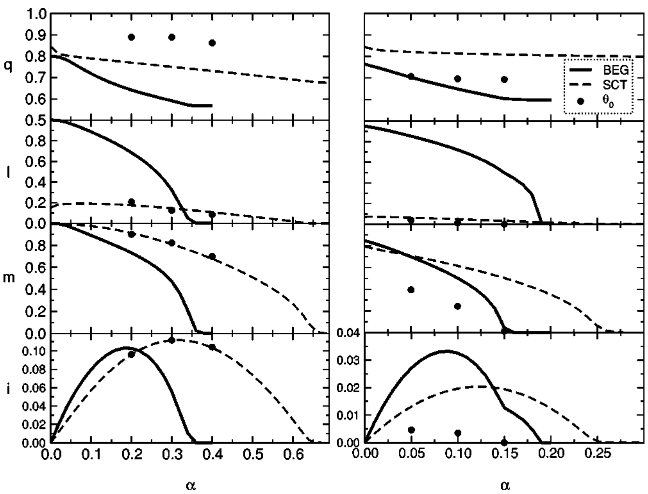

FIG. 2. The order parameters m, l, and q, and the information content i in the stationary state as functions of α for the BEG network, the SCT, and the optimal three-state networks, discussed in the text for $a=0.8$ and either $T=0.2$ (left) or $T=0.6$ (right).

also saddle-point solutions (s) for $m=0$, either with $l\neq0$ or with $l=0$ for Q and S, respectively, as indicated in the table. The saddle points have one-dimensional basins of attraction with attractor directions along l, either towards $l^*\neq0$ or to $l^*=0$, respectively, and repeller directions along m away from $m=0$.

Thus, there is a retrieval phase in regions I–V and, in contrast to earlier work [1], it is the only stable phase in regions I–III. The quadrupolar phase exists only at large T in regions VI and VII. The self-sustained activity phase is the only phase for large T and α. Full (dotted) lines denote discontinuous (continuous) transitions, heavy lines denote the boundary of the R phase. The lines at the most right yield the critical storage capacity α_c, where both overlaps m and l disappear. A similar behavior appears for other big values of the pattern activity a, whereas for small a there are only R and S phases. The reason for a low-T retrieval phase and the absence of a Q phase is that a finite T is needed for the active neurons (±1) to coincide with the active patterns but with uncorrelated signs, such that $m=0$. This can take place with a finite fluctuation overlap up to a higher synaptic T or stochastic α noise.

The typical α dependence of the order parameters and the information content below and above the threshold $(\alpha,T)=(0.22,0.45)$ where a stable Q phase starts to appear are shown in Fig. 2. Clearly, for T below that threshold (left figure) m and l remain finite together, in a behavior characteristic of retrieval, up to the critical α_c. In this regime the fluctuation overlap does not yield anything essentially new that is not contained in the retrieval overlap. In contrast, above the threshold (right figure) m disappears first with increasing α leaving a finite $l\neq0$ up to a bigger α_c. Hence, first T and then α have to become large enough for the Q states to appear. Note that the fluctuation overlap carries a finite information even with $m=0$ in the Q phase. Thus, although the information transmitted by the network is mainly in the retrieval phase, there is also some information

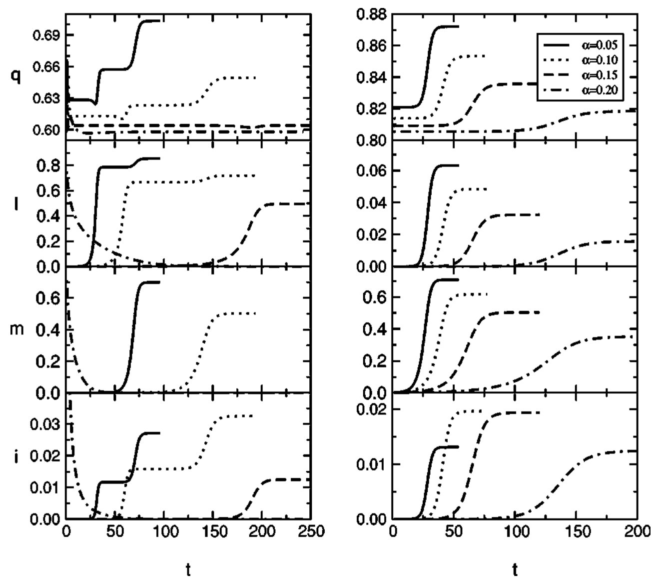

FIG. 3. Time evolution of the order parameters m, l, and q and information content i, for $a=0.8$, $T=0.6$, and α, as indicated. The BEG network (left) and the SCT network (right).

due to the Q phase. This information is provided by the fact that the active neurons coincide with the active patterns but the signs are not correlated. An example of its practical use in pattern recognition is given in the beginning. We also show in Fig. 2 the comparison of the performance with two other three-state networks. The first network is the usual network with an externally adjustable optimal threshold parameter given by a uniform $\theta=\theta_i$ [7,8] that yields the largest mutual information (dotted lines).

The second network is a phenomenological extension to finite T of a recent three-state self-control model (SCT) [6,10], (dashed lines) in which the self-control threshold θ_t at $T=0$ is replaced by a linearly shifted threshold $\overline{\theta_t}=\theta_t-T$, where $\theta_t=\sqrt{2\ln a}\,D_t$ with $D_t^2=\alpha q_t/a$ being the variance of the noise. As one of the main features of the BEG network, the results in Fig. 2 clearly show that in most of the regime where the retrieval overlap is nonzero the BEG network yields a much higher information content than the optimal threshold network and also a comparable or higher information than the SCT network for high T.

The time evolution of the order parameters and the information content are shown in Fig. 3 for $a=0.8$ and $T=0.6$ in both the BEG network (left) and in the SCT network (right). In support of the phase diagram shown in Fig. 1, one has first the asymptotic states of the R phase, then the states of a Q phase and, finally, the states of the S phase, in the BEG network with increasing α. In contrast, for the optimal and the SCT networks one has only the R phase up to the critical α_c. A closer examination of the curves for the BEG network reveals that, for small α, the fluctuation overlap "drives" a vanishingly small initial retrieval overlap, meaning almost no recognition of a given pattern by the network, into an asymptotic state with finite recognition. This is in contrast with the results for other three-state networks, as the SCT network, where first the overlap m_t becomes nonzero: m_t drives l_t. It is also worth noting that, with a vanishing initial m_0, the states of the network pass through the vicinity of a saddle point Q with a finite fluctuation overlap l and still a vanishing retrieval overlap at small or intermediate times. This is described by the first plateaus in q, l, and i. It is only in passing beyond those plateaus, which may take a rather long time, that the states attain the asymptotic behavior of the retrieval phase. With the initial conditions used for the BEG network, in the left part of the figure, there is no retrieval in the SCT network, meaning that the basins of attraction for retrieval are larger in the BEG network. Finally, the results for the dynamics and the stationary states are confirmed by flow diagrams in extension to previous work [11].

To summarize, we have studied the stability of a new Q phase in an extremely diluted BEG network and the role of the fluctuation overlap characteristic of this phase in enhancing the retrieval performance and the information content in this network over those of other three-state networks. We have found that the Q states are not stable neither at $T=0$ nor at higher T below a threshold, in contrast to earlier work and, hence, new phase diagrams have been obtained.

We have shown that the fluctuation overlap can drive a vanishingly small initial retrieval overlap to a large stationary value. We have also found that the dynamics may be slowed down due to the presence of saddle-point solutions that appear in large regions of the phase diagram, within the retrieval phase and close to the critical phase boundary. A possible role in practical applications of these information carrying Q states might be advocated.

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